2017

Full Marks - 80

Time - 3 hours

The figures in the right-hand margin indicate marks

Answer ***all*** questions

1. a) If H is a non-empty finite subset of a group G and H is closed under multipication, then prove that H is a subgroup of G. 5

b) Prove that if G is an abelian group, then for all $a, b \in G$ and all integers n, $(a \, . \, b)^n = a^n . n^n$. 5

c) i) If H and K are subgroups of G, show that $H \cap K$ is a subgroup of G. 3

ii) Let $G = \{1, w, w^2\}$, where w is a cube root of unity. Then prove that G is a group under multiplication. 3

OR

d) Prove that a nonempty subset H of the group G is a subgroup of G if and only if

i) $a, b \in H$ implies that $ab \in H$.

ii) $a \in H$ implies that $a^{-1} \in H$. 5

e) Let G be the set of all real 2 × 2 matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ where $a \neq 0$. Prove that G is an abelian group under multiplication. 5

f) i) Show that if every element of the group G is its own inverse, then G is abelian. 3

ii) Let G be the group of all real 2 × 2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc \neq 0$ under matrix multiplication. Let

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid ad \neq 0 \right\}$$

Prove that H is a subgroup of G. 3

2. a) The centre Z of a group G is defined by

$$Z = \{ z \in G / zx = xz, \text{ all } x \in G \}.$$

Prove that Z is a subgroup of G. 5

nonzero real numbers under multiplication. Define $\phi : G \to \overline{G}$ by

$$\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

verify ϕ is a homomorphism of G onto $\overline{G}$. 3

ii) Let G be any group, g a fixed element in G. Define $\phi : G \to G$ by $\phi(x) = gxg^{-1}$. Prove that ϕ is an isomorphism of G onto G. 3

□□

5. a) Let ϕ be a homomorphism of G onto $\overline{G}$ with Kernel K. Then prove that $\frac{G}{K} \approx \overline{G}$. 10

b) i) If ϕ is a homomorphism of G into $\overline{G}$, then prove that

a) $\phi(e) = \overline{e}$, the unit element of $\overline{G}$

b) $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$. 3

ii) Let G is the group of nonzero real numbers under multiplication, $\overline{G} = G$. $\phi(x) = x^2$ all $x \in G$. Prove that ϕ is homomorphism. Determine the Kernel. 3

OR

c) Prove that every group is isomorphic to a subgroup of A(S) for some appropriate S. 10

d) i) Let G be the group of all real 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0$, under matrix multiplication. Let $\overline{G}$ be the group of all

b) If G is a ginite group whose order is a prime number p, then prove that G is a cyclic group. 5

c) i) If H is a subgroup of G and $a \in G$, let

$$aHa^{-1} = \{aha^{-1} / h \in H\}.$$

Show that aHa^{-1} is a subgroup of G. 3

ii) Define a cyclic group and given an example of cyclic group. 3

OR

d) If H and K are finite subgroup of G, of order 0(H) and 0(K), respectively, then prove that

$$0(HK) = \frac{0(H)0(K)}{0(H \cap K)}$$ 10

e) i) Prove that any subgroup of a cyclic group is itself a cyclic group. 3

ii) If G is a finite group and $a \in G$, then prove that $a^{0(G)} = e$. 3

3. a) State and prove Lagrange's theorem. Give an example that its converse is not true. 10

b) i) Determine whether the following permutation is odd or even.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}$$ 3

ii) Prove that for all $a \in G$,

$$Ha = \{x \in G | a \equiv x \bmod H\}$$ 3

OR

c) Prove that if HK is a subgroup of G if and only if HK = KH. 10

d) i) Prove that permutation is the product of its cycles. 3

ii) Prove that if p is a prime number and a is any integer, then $a^p \equiv a \bmod p$. 3

4. a) Define normal subgroup of a group. Prove that the subgroup N of G is a normal subgroup of G if and only if every left coset of N in G is a right coset of N in G. 10

b) i) Show that every subgroup of an abelian group if normal. 3

ii) If G is a finite group and N is a normal subgroup of G, then prove that

$$0\left(\frac{G}{N}\right) = \frac{0(G)}{0(N)}$$ 3

OR

c) State and prove Cauchy's Theorem for abelian group. 10

d) i) If G is a group and H is a subgroup of index 2 in G. Prove that H is a normal subgroup of G. 3

ii) Prove that N is a normal subgroup of G if and only if $gNg^{-1} = N$ for every $g \in G$. 3