2019

Full Marks - 80

Time - 3 hours

The figures in the right-hand margin indicate marks.

Answer ***all*** questions

1. a) Prove that an order field F has least upper bound property if and only if it has greatest lower bound property. 8

   b) Let $x > 1$. Then prove that the set $S = \{x^n : n \in \mathbb{N}\}$ is unbounded above. 4

   c) Find the maximum, minimum, supremum and infimum of the sets, if exists. 4

   i) $\left\{\frac{1}{m}+\frac{1}{n} : m, n \in \mathbb{N}\right\}$

   ii) $\left\{(-1)^n\left(1+\frac{1}{n}\right) : n \in \mathbb{N}\right\}$

OR

d) Prove that Q is not a complete order field. 8

e) Let $S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$. Find spremum, infimum, maximum and minimum of S, if exists. 4

f) Define bounded and unbounded set. Give example of sets 4

i) for which lub = glb.

ii) for which lub does not exist and glb belongs to the set.

iii) which does not contain its lub and glb.

2\. a) State and prove Archinedean, property. 8

b) Show that Q is dense in R. 4

c) Give examples of 4

i) an infinite set with no limit point

ii) A bounded set with no limit point.

iii) An unbounded set with no limit point.

e) Test the conditional and absolute converagence of the series. 4

i) $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + ...$

ii) $1 - \left(\frac{1}{3}\right) + \left(\frac{1}{5}\right) - \left(\frac{1}{7}\right) + ...$

f) Show that the series 4

$$\sum \sqrt{n^2+1} - \sqrt{n^2-1}$$

f) Show that the harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. 4

5. a) Let $a_n \in \mathbb{C}$. If $\sum a_n$ is absolutely convergent then prove that it is convergent but converse is not true. 8

b) Test the conditional and absolute convergence of the series 4

i) $\sum \frac{(-1)^{n-1}}{n}$

ii) $\sum \frac{(-1)^n}{\log^n}$.

c) If $\sum a_n$ is absolutely convergent and (b_n) is bounded, then prove that $\sum a_n b_n$ converge absolutely. 4

OR

d) If $\sum a_n$ is absolutely convergent, then prove that any rearrangement of $\sum a_n$ has the same sum. 8

iv) An unbounded set with exactly one limit point.

OR

d) State and prove rational density theorem. 8

e) If $a, b \in R_+$ such that $a < b$ then show that there exist a rational number $\frac{K}{10^n}$ where $K \in N, n \in \mathbb{N}$ s.t. $a < b - \frac{1}{10^n} \leq \frac{K}{10^n} < b$. 4

f) Find the limit point of the sets. 4

i) $S = \{ (-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$

ii) $T = (0, 1)$

3. a) Prove that a sequence of real numbers is convergent iff it is Cauchy sequence. 8

b) Show that 4

i) $\lim \sqrt{n^2+1} - n = 0$

ii) $\lim(x^{1/n}) = 1$ if $x > 0$

c) If (x_n), (y_n), (z_n) are three sequences

s.t. i) $x_n \le y_n \le z_n \ \forall n \in \mathbb{N}$

(ii) $\lim x_n = \lim z_n = l$

Then prove that $\lim y_n = l$. 4

OR

d) State and prove Bolzano Weirstrass theorem for sequence. 8

e) i) Give example of two sequences (x_n) and (y_n) both divergent but $(x_n + y_n)$ is convergent. 4

ii) Determine a monotone subsequence of the sequence $\left(\frac{1}{n}\right)$.

f) Prove that every convergent sequence is bounded. 4

4. a) Let $\sum b_n$ be conveges. If there exist a consetant $K > 0$ such that $0 \le a_n \le kb_n$, then prove that $\sum a_n$ is convergent. 8

b) Test the convergence of the series 4

i) $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$

ii) $\sum_{n=1}^{\infty} \left(\frac{n}{3^n}\right)$

c) Show that the series $\sum_{n=1}^{\infty} \frac{n+5}{n(n+1)\sqrt{(n+2)}}$ converges. 4

OR

d) Show that the series $\sum \frac{1}{n!}$ converges to e, where e is an irrational number. 8

e) Test the convergence of the series 4

i) $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

ii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$